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ABSTRACT. We show that for odd-valued piecewise-constant skew products over a certain two parameter family of interval exchanges, the skew product is ergodic for a full-measure choice of parameters.

1. INTRODUCTION AND BACKGROUND

$\mathbb{Z}$-valued (or more generally G-valued where G is a locally compact group) skew products are a natural construction of infinite-measure preserving transformations using ergodic sums over a finite-measure preserving transformation. For a thorough overview of constructing skew products over irrational rotations, see [3]. The natural generalization of an irrational rotation is an *interval exchange transformation*; recent work in studying generic skew products over generic interval exchanges may be found in [1], where the authors establish ergodicity of skew products for step functions over generic interval exchanges. We present here an alternate 'hands-on' approach to prove generic ergodicity for one specific construction.

Let $X = \mathbb{S}^1 \times \{0, 1, \ldots, k-1\}$, endowed with Lebesgue measure μ (scaled so $\mu(X) = k$), and assume that $k = 1 \bmod 2$. Let T be a map on X defined by

$$T(x, \ell) = \big((x+\alpha) \bmod 1, (\ell + I(x)) \bmod k\big), \tag{1}$$

where $I(x)$ is the characteristic function of an interval of length β, and α is irrational; $\{X, T\}$ is a $\mathbb{Z}/k\mathbb{Z}$-valued skew product (in fact a cyclic extension) of the irrational rotation by α. Let f be an integer-valued function on X. The skew products we will consider are given by

$$T_f(x, \ell, m) = \big((x+\alpha) \bmod 1, (\ell + I(x)) \bmod k, m + f(x, \ell)\big).$$

Denote by $S_m(x, \ell)$ the $\mathbb{Z}$-coordinate of $T_f^m(x, \ell, 0)$:

$$S_m(x, \ell) = \sum_{i=0}^{m-1} f(T^i(x, \ell)).$$

Note that $\{X \times \mathbb{Z}, T_f\}$ will *not* in general itself be a skew product over rotation by α, as $f(x, \ell)$ is not independent of ℓ. We assume that f is of mean zero, and assume further that f is piecewise constant on finitely many intervals; let $\mathrm{Var}(f)$ be the sum over ℓ of the (finite) variations of f restricted to each $\mathbb{S}^1 \times \{\ell\}$. Purely

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for convenience we furthermore assume that I and f are right-continuous; they are defined using intervals closed on the left and open on the right.

An integer E is an *essential value* of our skew product if for every $A \subset X$ of positive measure, there is some i such that

$$\mu\left(A \cap T^i A \cap \{(x,\ell) : S_i(x,\ell) = E\}\right) > 0.$$

If E is an essential value, the skew product is ergodic if and only if the skew product given by f into $\mathbb{Z}/(E\mathbb{Z})$ is ergodic.

We will use *Koksma's inequality*: let P be a partition of $\mathbb{S}^1$ into q intervals of equal length, let f be real-valued, of bounded variation on $\mathbb{S}^1$, and suppose that x_1 through x_n are chosen such that each interval of P contains exactly one x_m. Then

$$\left|\sum_{m=1}^{n} f(x_m) - n\int_{\mathbb{S}^1} f(x)dx\right| \leq \mathrm{Var}(f).$$

Our interval exchanges are characterized by two choices: α and β.

Theorem 1.1. *Let f take only odd values, and assume that not every value of f is a multiple of the same number. Then the set of α, β for which the skew product is ergodic is of full measure.*

2. Proof

Lemma 2.1. *Let f take integer values (not necessarily odd) and assume that not every value of f is a multiple of the same number. Further let $\beta \in (0,1)$ be fixed, and assume there is some finite, nonzero $E \in \mathbb{Z}$ which is an essential value of the skew product $\{X \times \mathbb{Z}, T_f\}$. Then the set of α for which the skew product is ergodic is of full measure.*

Proof. Suppose that β is fixed and not zero. We can construct a compact, connected translation surface M and a cross-section X so that the the first return map to X of the geodesic flow in the direction with slope $1/\alpha$ is T given by (1) for the parameters α, β.

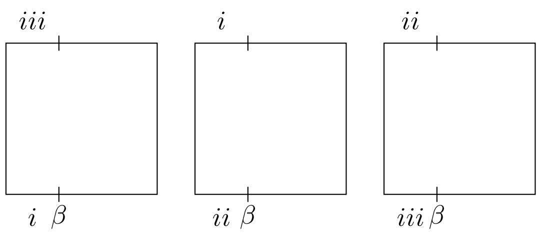

FIGURE 1. The translation surface M for $k = 3$ and $I(s) = 1_{[0,\beta)}$. The unlabeled sides are identifies to the opposite side in the same square, the other identifications are given by roman numbers. The cross-section $X \times \{0,1,2\}$ consists of the bottom of the three squares. The flow in the vertical direction corresponds to $\alpha = 0$.

By [4], the system $\{X, \mu, T\}$ is (uniquely) ergodic for almost every choice of α. Now let $X' = X \times \{0, 1, \ldots, E-1\}$, with the identification

$$(x, \ell, k) \sim (x, \ell, k + f(x,\ell) \mod E)$$

for each $(x, \ell) \in X$. This identification corresponds to gluing together E disjoint copies of M via the values given by f, taken modulo E; denote this new surface by M'. So long as M' is connected, the results of [4] still apply, and the transformation

$$S'(x, \ell, k) = (x + \alpha \mod 1, \ell + I(x), k + f(x, \ell) \mod E)$$

is uniquely ergodic for almost every choice of α. The assumption that the values of f generate $\mathbb{Z}$ exactly ensure that M' is connected via Bézout's Lemma: the values taken by f on each $X \times \{j\}$ do not depend on the choice of $j \in \{0, 1, \ldots, E-1\}$, and there is no single common divisor for the set of values taken by f, so we may freely pass from one copy of M to another via the values of f to generate any integer value. Ergodicity of the skew product for each α such that this finite system is ergodic then follows as E was assumed to be an essential value of $\{X \times \mathbb{Z}, \mu \times dz, T_f\}$. $\square$

The effect of Lemma 2.1 is to reduce our problem to the existence of a single nonzero, finite essential value for generic choice of β. We now re-introduce the assumption that the values of f are all odd (and still not multiples of the same number). Let α be irrational with continued fraction expansion

$$\alpha = [a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where each a_m is a positive integer; an excellent reference for the theory of continued fractions is [5]. Denote by p_n/q_n the convergents to α, and by $\|\cdot\|$ the distance to the nearest integer. Then it is well-known that

$$q_n \|q_n \alpha\| \leq \frac{1}{a_{n+1}}. \tag{2}$$

On X we also use $\|\cdot\|$ for distance, with the convention that if $\ell \neq \ell'$, $\|(x, \ell) - (y, \ell')\| = 1$. We denote by $Q_n(T)$ the periodic approximation to T given by

$$Q_n(x, \ell) = \left(x + \frac{p_n}{q_n} \bmod 1, \ell + I(x) \bmod k \right).$$

Definition 2.2. A point $x \in X$ will be called *n-good for rational approximation* if for all $i = 0, 1, \ldots, kq_n - 1$ we have

$$f(T^i x) = f(Q_n^i(x)), \quad I(T^i x) = I(Q_n^i x).$$

That is, as far as the functions f and I are concerned, through time kq_n we may replace the orbit of x under T with the orbit of x under Q_n.

Definition 2.3. A point $x \in X$ will be called *n-spread out* if the set $\{T^i(x)\}$, $i = 0, 1, \ldots, kq_n - 1$, has the property that

- there are exactly q_n points in each $\mathbb{S}^1 \times \{\ell\}$, and
- for each ℓ, there is a partition of $\mathbb{S}^1 \times \{\ell\}$ into disjoint intervals of length $1/q_n$ such that there is exactly one of the $T^i x$ in each partition element.

Lemma 2.4. *Suppose that x is n-spread out. Then*

$$\left| \sum_{i=0}^{kq_n - 1} f(T^i x) \right| \leq \mathrm{Var}(f).$$

Proof. The restriction of the orbit of x to each $\mathbb{S}^1 \times \{\ell\}$ may be summed separately, and the n-spread out assumption allows us to use Koksma's inequality on each $\mathbb{S}^1 \times \{\ell\}$. $\square$

Let $D = \{d_1, \ldots, d_N\}$ be the projection of all discontinuities of f onto $\mathbb{S}^1$ together with the discontinuities of $I(x)$. For $n = 0 \bmod 2$ define

$$A_n = \left(\mathbb{S}^1 \setminus \left(\bigcup_{i=0}^{kq_n-1} \bigcup_{j=1}^{N} \Big[d_j - k\|q_n\alpha\| - i\alpha, d_j - i\alpha\Big)\right)\right) \times \{1, 2, \ldots, k\},$$

while for $n = 1 \bmod 2$ we use the intervals

$$\Big(d_j - i\alpha, d_j + k\|q_n\alpha\| - i\alpha\Big].$$

Lemma 2.5. *Each $x \in A_n$ is n-good for rational approximation, and*

$$\mu(A_n) \geq k\left(1 - k^2 N q_n \|q_n\alpha\|\right) \geq k\left(1 - \frac{k^2 N}{a_{n+1}}\right).$$

Proof. The first inequality is elementary (assume all removed intervals are disjoint), and the final inequality is simply due to (2); the only content to prove is that $x \in A_n$ implies that x is n-good for rational approximation. Suppose that $n = 0 \bmod 2$ so that $p_n/q_n > \alpha$. Let $x \in A_n$; there is no $i < kq_n$ such that

$$x + i\alpha \in [d_j - k\|q_n\alpha\|, d_j).$$

The distance between $x + i\alpha$ and $x + ip_n/q_n$ is no larger than $k\|q_n\alpha\|$, so we cannot have

$$x + i\alpha < d_j \leq x + i\frac{p_n}{q_n}$$

for any i, j. As $p_n/q_n > \alpha$, this completes the proof for $n = 0 \bmod 2$. For $n = 1 \bmod 2$ the process is identical, but we remove intervals from the other side of the discontinuities d_j, and $p_n/q_n < \alpha$. $\square$

Definition 2.6. The action of T^{kq_n} on A is *nearly-rigid* if $\|x - T^{kq_n}(x)\| \leq k\|q_n\alpha\|$ for all $x \in A$.

Lemma 2.7. *The action of T^{kq_n} on A_n is nearly-rigid.*

Proof. Through time q_n the point x orbits into the interval defining $I(x)$ some number of times. Under Q_n, however, x has returned exactly to the same $\mathbb{S}^1$ coordinate. Over the next q_n times, the orbit of x will therefore intersect this interval *the same number of times* (recall that $I(x, \ell)$ is independent of ℓ), and so on for each q_n steps in the orbit. Whatever this number of intersections is, once we have applied Q_n a total of kq_n times, the total number of points in these intervals must be zero modulo k: $Q_n^{kq_n}(x) = x$. As $x \in A_n$, we certainly have $T^{kq_n}(x)$ belonging to the same copy of $\mathbb{S}^1$ as x, then, and the distance in $\mathbb{S}^1$ between x and $T^{kq_n}(x)$ is equal to $\|kq_n\alpha\|$, which is no larger than $k\|q_n\alpha\|$. $\square$

Definition 2.8. The set A is *nearly invariant* under T if

$$\mu(A \triangle T(A)) \leq 2k^2 N\|q_n\alpha\|.$$

Lemma 2.9. *The set A_n is nearly invariant under T.*

Proof. Recall that A_n is constructed by removing successive preimages of kN different intervals of length $k\|q_n\alpha\|$ (N such intervals in each copy of $\mathbb{S}^1$). Therefore $A_n \triangle T(A_n)$ at most consists of the first image of these intervals and the next preimage. □

Define

$$\sigma_n(x) = \sum_{i=0}^{q_n-1} I\left(x + \frac{i}{q_n} \bmod 1\right).$$

Note that if $x \in A_n$, then

$$\sigma_n(x) = \sum_{i=0}^{q_n-1} I(T^i x).$$

Lemma 2.10. *If $x \in A_n$, $a_{n+1} \geq k$, and $\sigma_n(x)$ is relatively prime to k, then x is n-spread out.*

Proof. Note that $\sigma_n(x)$ is exactly the number of times through time q_n that $I(Q_n^i x) = 1$. By the assumption that $x \in A_n$, this is also the number of times that $T^i x$ will orbit into this interval, and furthermore this number will be repeated for each successive length-q_n segment of the orbit we consider:

$$x \in A_n \quad \Longrightarrow \quad \sigma_n(x) = \sigma_n(T^{q_n} x) = \ldots = \sigma_n(T^{(k-1)q_n} x).$$

As $\sigma_n(x)$ was assumed to be relatively prime to k (i.e. $\sigma_n(x)$ generates $\mathbb{Z}/k\mathbb{Z}$), it follows that for each $i = 0, 1, \ldots, q_n - 1$, each of

$$\{T^{i+\ell q_n}(x)\} \quad (\ell = 0, 1, \ldots, k-1)$$

belongs to a *different* copy of $\mathbb{S}^1$. Finally, the assumption that $a_{n+1} \geq k$ implies (again via (2)) that

$$k\|q_n\alpha\| < \frac{1}{q_n},$$

so the intervals $[x + i/q_n, x + (i+1)/q_n)$ in each circle (if $n = 0 \bmod 2$; for $n = 1 \bmod 2$ reverse which end is closed versus open) each contain one element of the orbit. □

Lemma 2.11. *For all x, $\sigma_n(x) \in \{M, M+1\}$, where $M = [q_n\beta]$, the integer part of $q_n\beta$.*

Proof. The number M is the minimum number of abutting intervals of length $1/q_n$ (closed on the left, open on the right, say) which will always be completely contained within an interval of length β:

$$\frac{M}{q_n} \leq \beta < \frac{M+1}{q_n}.$$

For any x, then, there are at least M successive $I(x + i/q_n) = 1$. On the other hand, as $(M+1)/q_n > \beta$, no x may have $\sigma_n(x) \geq M + 2$. □

Definition 2.12. If T^{kq_n} is nearly rigid and there is some $\epsilon > 0$ such that $\mu(A_n) \geq \epsilon$ then T is called *quasi-rigid* and the A_n are called *quasi-rigidity sets.*

Corollary 2.13. *Suppose that for infinitely many n we have*

- $a_{n+1} > k^2 N$,
- $q_n = 1 \bmod 2$,
- *$\sigma_n(x)$ is relatively prime to k for all $x \in X$.*

Then there is a finite nonzero essential value.

Proof. The assumption that $a_{n+1} > k^2N$ implies that the A_n are quasi-rigidity sets (via Lemmas 2.5 and 2.7). That $\sigma_n(x)$ is relatively prime to k ensures that for each $x \in A_n$, x is n-spread out, so by applying the Koksma inequality there is a uniform bound on the absolute value of the ergodic sums on A_n. We therefore apply [2, Corollary 2.6] (utilizing that the A_n are quasi-rigid and nearly invariant, which we have already established) to find an essential value (possibly zero) for the skew product; in short, as there is an upper bound on the sums from Koksma's inequality, we may pass to a sequence of subsets along which a single value is seen, and this value is therefore an essential value. As kq_n is odd and f takes only odd values, it follows that for all $x \in A_n$ we must have

$$\left| \sum_{i=0}^{kq_n-1} f(T_f^i(x)) \right| \geq 1,$$

so therefore the essential value we have found in this manner is not zero. □

It is therefore of interest to determine when $\sigma_n(x)$ is relatively prime to k.

Lemma 2.14. *Let $\{m_i\}$ be an unbounded sequence of integers, and let k be a positive integer. Then for each residue class $j \mod k$, for almost every θ the equality*

$$[m_i\theta] = j \mod k$$

is satisfied for infinitely many i.

Proof. Without loss of generality, assume that $\{m_i\}$ are unbounded above, and by passing to a subsequence, we may assume that the m_i are *superlacunary*:

$$\lim_{i\to\infty} \frac{m_{i+1}}{m_i} = \infty.$$

Also, without loss of generality assume $\theta \in [0, 1]$, and define the random variable

$$X_i(\theta) = [m_i\theta] \mod k.$$

Suppose that $X_{i-1}(\theta) = R$, so that for some M we have

$$\theta = \frac{R + Mk}{m_{i-1}} + \frac{\{m_{i-1}\theta\}}{m_{i-1}},$$

where $\{x\}$ denotes the fractional part of x. The residue class of $[m_i\theta]$, then, is determined by the residue class of R', where

$$\theta \in \left[\frac{R'}{m_i}, \frac{R'+1}{m_i}\right).$$

As the $\{m_i\}$ are superlacunary, the number of intervals of length $1/m_i$ within an interval of length $1/m_{i-1}$ diverges, from which it follows that

$$\lim_{i\to\infty} \mathbb{P}\left(X_{i+1} = j | X_i\right) = \frac{1}{k}$$

for each residue class j. So along this superlacunary subsequence, for generic θ the sequence $[m_i\theta]$ is uniformly distributed among the residue classes, from which the lemma trivially follows. □

Corollary 2.15. *For almost every choice of α, β, there are infinitely many n such that such that $a_{n+1} > k^2N$, $q_n = 1 \bmod 2$, and $[q_n\beta] = 1 \bmod k$.*

Proof. For generic α there are infinitely many pairs a_{n+1}, a_{n+2} of arbitrarily large partial quotients, and no two consecutive q_n, q_{n+1} may be even, so the first two conditions are trivially satisfied. The $\{q_n\}$ are an increasing sequence of integers, so by Lemma 2.14, for almost every β arbitrary residue classes of $[t_m\beta]$ modulo any fixed k are achieved infinitely many times.

□

This completes the proof of ergodicity: for generic choice of α, β the skew product will have a nonzero essential value E by Corollary 2.13 (as k is odd, both one and two are relatively prime to k). By Lemma 2.1, this suffices for generic ergodicity.

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